GIRRAWEEN HIGH SCHOOL

HALF YEARLY EXAMINATION

YEAR 12

2011

MATHEMATICS

EXTENSION 1

Time allowed – Two hours

(Plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- All necessary working should be shown in every question.
  Marks may be deducted for careless or badly arranged work.
- Board-approved calculators may be used.
- Each question attempted is to be returned on a *separate* piece of paper clearly marked Question 1, Question 2, etc. Each piece of paper must show your name.
- You may ask for extra pieces of paper if you need them.

Total Marks – 100

Attempt all questions 1-5

All questions are NOT of equal value.

Answer each question clearly ON A SEPARATE PAGE!

Question 1 (20 Marks) Use a separate piece of paper. **Marks**

(a) Find the ratio in which the point (2,5) divides the interval AB where A= (4, 9), B= (-3, -5). **3**

(b) Expand and simplify $\cos(x-y)-\cos(x+y)$. **2**

(c) Find an expression for $\sqrt{\dfrac{1+\cos\theta}{1-\cos\theta}}$, in terms of t, $t=\tan\dfrac{\theta}{2}$. **3**

(d) Solve $\dfrac{2x-6}{x}<1$. **4**

(e) If α, β and γ are the roots of the equation $x^3+2x^2+3x+4=0$, find the value of:

(i) $(\alpha-1)(\beta-1)(\gamma-1)$ **2**

(ii) $\dfrac{1}{\alpha}+\dfrac{1}{\beta}+\dfrac{1}{\gamma}$ **2**

(iii) $\dfrac{1}{\alpha\beta}+\dfrac{1}{\alpha\gamma}+\dfrac{1}{\beta\gamma}$ **2**

(f) Find the coefficient of x^3 in the expansion $(x-5)^4$. **2**

Question 2 (27 Marks) Use a separate piece of paper.

(a) Find the number of six-letter arrangements that can be made from the letters in the word

SYDNEY. **2**

(b) Differentiate:

(i) $\dfrac{e^x + 1}{2x}$ **3**

(ii) $\ln\left(\dfrac{x}{x^2 + 1}\right)$ **3**

(c) Evaluate:

(i) $\displaystyle\int_0^1 x^2 e^{-x^3}\,dx$ **3**

(ii) $\displaystyle\int_0^1 2^x\,dx$ **3**

(d) The gradient of a curve is given by $y' = e^{2-x}$ and the curve passes through the point (0,1). Find the equation of the curve and its horizontal asymptote. **5**

(e) (i) Find $\dfrac{d}{dx}\left(e^x + e^{-x} \right)$. **1**

(ii) Hence, find $\displaystyle\int_0^2 \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right) dx$ **3**

(f) Find the equation of the normal to $y = e^{-x}$ at the point P (-1, e). **4**

Question 3 (17 Marks) Use a separate piece of paper.

(a) When Amy crossed a tall strain of pea with a dwarf strain of pea, she found that $\frac{3}{4}$ of the offspring were tall and $\frac{1}{4}$ were dwarf. Suppose five such offspring were selected at random. Find the probability that:

(i) All of these offspring were tall. **2**

(ii) At least three of these offspring were tall. **3**

(b) Solve the equation $2x^3 - 7x^2 - 12x + 45 = 0$ given that two of its roots are equal. **4**

(c) Given that $(x-3)$ and $(x+2)$ are factors of $x^3 - 6x^2 + px + q$, find the values of p and q. **4**

(d) Prove by mathematical induction that for $n \geq 1$

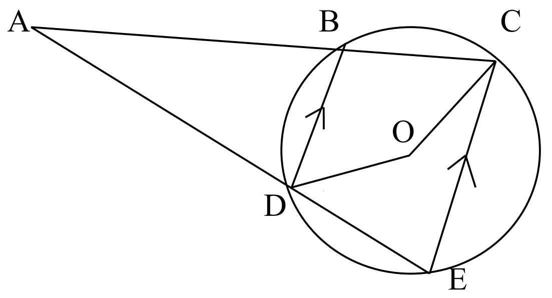
$$1^2 + 3^2 + \ldots\ldots + (2n-1)^2 = \frac{1}{3}n(2n-1)(2n+1).$$ **4**

Question 4 (11 Marks) Use a separate piece of paper.

(a) Find the acute angle between the lines $x - 2y = 6$ and $4x - y = 1$. **3**

(b) Prove that $\dfrac{1-\cos 2\theta}{\sin 2\theta} = \tan\theta$. **2**

(c)



BD and CE are two parallel chords of a circle with centre O. CB and ED produced meet at A. Prove that:

(i) AC=AE **3**

(ii) ACOD is a cyclic quadrilateral. **3**

Question 5 (25 Marks) Use a separate piece of paper.

(a) $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are two points on the parabola $x^2 = 4ay$.

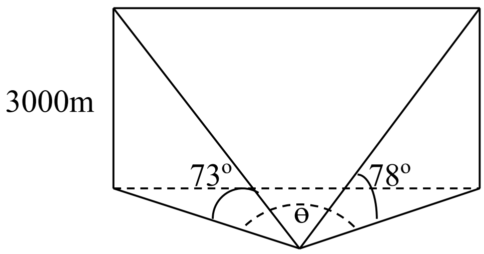
(i) If PQ passes through the point $R(2a, 3a)$ show that $pq = p + q - 3$. **5**

(ii) If M is the midpoint of PQ, show that the coordinates of M are:

$$\left[\ a(pq+3),\quad \frac{a}{2}(pq+3)^2 - 2pq\right]$$ **4**

(iii) Hence, find the locus of M. **3**

(b) A hot air balloon flying at 950m/h at a constant altitude of 3000m is observed to have an angle of elevation of 78°. After 20 minutes, the angle of elevation is 73°. Calculate the angle through which the observer has turned during those 20 minutes. **4**



(c) Write $\sqrt{3}\sin x - \cos x$ in the form, $r\sin(x - \alpha)$. **4**

(d) Solve using the t-formula $4\sin\theta - 3\cos\theta = 2$ for $0^o \le \theta \le 360^o$. **5**

END OF PAPER.

Yr 12 HY EXT 1 2011.

Question 1.

a) A $(4,9)$ B $(-3,-5)$

$k : 1$ $P(2,5)$

$$2 = \frac{1(4)+k(-3)}{1+k}$$

$$2+2k = 4-3k$$
$$5k = 2$$
$$k = \frac{2}{5}$$

b) $\cos(x-y) - \cos(x+y)$

$= [\cos x \cos y + \sin x \sin y] - [\cos x \cos y - \sin x \sin y]$

$= \cos x \cos y + \sin x \sin y - \cos x \cos y + \sin x \sin y$

$= 2 \sin x \sin y$

c) $\sqrt{\dfrac{1+\cos\theta}{1-\cos\theta}}$

$= \sqrt{\dfrac{1+\left(\frac{1-t^2}{1+t^2}\right)}{1-\left(\frac{1-t^2}{1+t^2}\right)}}$

$= \sqrt{\dfrac{\frac{1+t^2+1-t^2}{1+t^2}}{\frac{1+t^2-1+t^2}{1+t^2}}}$

$= \sqrt{\dfrac{2}{2t^2}}$

$= \sqrt{\dfrac{1}{t^2}} = \dfrac{1}{t}$

d) By critical points.

$$\frac{2x-6}{x} < 1.$$

Equality

$$\frac{2x-6}{x} = 1$$
$$2x-6 = x$$
$$x = 6$$

Discontinuity

$$x = 0$$

Testing.

$x=-10$ $x=1$ 6 $x=7$

$\dfrac{-2-6}{-1}$ $\dfrac{2-6}{1}$ $\dfrac{14-6}{7}$

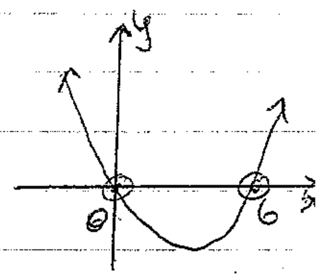
$4 \not< 1$ $-4<1$ $\frac{8}{7} \not< 1$

False True False

$\therefore 0 < x < 6$

OR $\dfrac{2x-6}{x} < 1$

$$\frac{x^2(2x-6)}{x} < x^2$$
$$2x^2 - 6x < x^2$$
$$x^2 - 6x < 0$$
$$x(x-6) < 0$$



$\therefore 0 < x < 6.$

1 cont.

e) $x^3 + 2x^2 + 3x + 4 = 0.$

$\alpha + \beta + \gamma = -2.$

$\alpha\beta + \beta\gamma + \gamma\alpha = 3$

$\alpha\beta\gamma = -4.$

) $(\alpha - 1)(\beta - 1)(\gamma - 1)$

$= \alpha\beta\gamma - \alpha\beta - \alpha\gamma + \alpha - \beta\gamma + \beta + \gamma - 1$

$= \alpha\beta\gamma - (\alpha\beta + \alpha\gamma + \beta\gamma) + \alpha + \beta + \gamma - 1$

$= -4 - 3 - 2 - 1.$

$= -10.$

) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

$= \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$

$= -\frac{3}{4}$

ii) $\frac{1}{\alpha\beta} + \frac{1}{\alpha\gamma} + \frac{1}{\beta\gamma}$

$= \frac{\gamma + \beta + \alpha}{\alpha\beta\gamma}$

$= \frac{1}{2}.$

f) $(x-5)^4$

$= x^4 + 4x^3(-5) + 6x^2(-5)^2 + 4x(-5)^3 + (-5)^4$

$= x^4 - 20x^3 + 150x^2 - 600x + 625$

$\therefore$ The coefficient of x^3 is -20.

Question 2

a) No. of 6 letter arrangements

$= \frac{6!}{2!}$

$= \frac{1 \times 2 \times 3 \times 4 \times 5 \times 6}{1 \times 2}$

$= 360.$

bi). $\frac{d}{dx}\ \frac{e^x + 1}{2x}$

Let $u = e^x + 1.$

$u' = e^x.$

$v = 2x.$

$v' = 2$

$\frac{dy}{dx} = \frac{2x(e^x) - (e^x+1)2}{(2x)^2}.$

$= \frac{2xe^x - 2e^x - 2}{4x^2}.$

$= \frac{2(xe^x - e^x - 1)}{4x^2}.$

$= \frac{xe^x - e^x - 1}{2x^2}$

ii) $\frac{d}{dx}\ \ln\left(\frac{x}{x^2+1}\right)$

$= \frac{d}{dx}\ \ln x - \ln(x^2+1).$

$= \frac{1}{x} - \frac{2x}{x^2+1}.$

$= \frac{(x^2+1) - 2x^2}{x(x^2+1)}$

$= \frac{1 - x^2}{x(x^2+1)}$

Q2 cont.

c i) $\int_0^1 x^2 e^{-x^3}\,dx$

$= -\frac{1}{3}\int_0^1 -3x^2 e^{-x^3}\,dx$

$= -\frac{1}{3}\left[e^{-x^3}\right]_0^1$

$= -\frac{1}{3}\left(e^{-1} - e^{0}\right)$

$= \frac{1}{3} - \frac{1}{3e}$

$= \frac{e-1}{3e}$

ii) $\int_0^1 2^x\,dx = \int_0^1 e^{x\log 2}\,dx$

$= \left[\frac{1}{\log 2} e^{x\log 2}\right]_0^1$

$= \left[\frac{2^x}{\log 2}\right]_0^1$

$= \left[\frac{2}{\log 2} - \frac{1}{\log 2}\right] = \frac{1}{\log 2}$

d) $y' = e^{2-x}$

$\int e^{2-x}\,dx$

$= -\int -e^{2-x}\,dx$

$y = -e^{2-x} + C$

passes through $(0,1)$

$\therefore 1 = -e^2 + C$

$C = 1 + e^2$

$\therefore y = -e^{2-x} + 1 + e^2$

Horizontal Asymptote

$y = 1 + e^2$

e) i) $\frac{d}{dx}\, e^x + e^{-x}$

$= e^x - e^{-x}$

ii) $\int_0^2 \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$

$= \left[\ln\left(e^x + e^{-x}\right)\right]_0^2$

$= \ln\left(e^2 + e^{-2}\right) - \ln\left(e^0 + e^0\right)$

$= \ln\left(\frac{e^2 + e^{-2}}{2}\right)$

Q2 cont.

f) $y = e^{-x}$

$y' = -e^{-x}$ at $P(-1, e)$

$m_{tangent} = -e$.

$m_{normal} = \frac{1}{e}$

$\therefore$ Eqn$_{normal}$: $y - e = \frac{1}{e}(x+1)$

$$y - e = \frac{x}{e} + \frac{1}{e}$$

$$ey - e^2 = x + 1.$$

$\therefore x - ey + e^2 + 1 = 0$.

Question 3

a) $t = \frac{3}{4}$ $\quad d = \frac{1}{4}$.

i) $P(\text{all tall}) = {}^5C_5 \left(\frac{3}{4}\right)^5$

$$= \frac{243}{1024}$$

ii) $P(\text{at least 3 tall})$

$= {}^5C_3 \left(\frac{3}{4}\right)^3 \left(\frac{1}{4}\right)^2 + {}^5C_4 \left(\frac{3}{4}\right)^4 \left(\frac{1}{4}\right)$

$+ {}^5C_5 \left(\frac{3}{4}\right)^5$

$= 10\left(\frac{27}{4^5}\right) + 5\left(\frac{3^4}{4^5}\right) + \left(\frac{3^5}{4^5}\right)$

$= \frac{34(3^3)}{4^5}$.

b) $2x^3 - 7x^2 - 12x + 45 = 0$

Roots α, α, β.

$\alpha + \alpha + \beta = \frac{7}{2}$.

$2\alpha + \beta = \frac{7}{2}$ (1)

$\alpha\alpha\beta = -\frac{45}{2}$.

$\alpha^2\beta = -\frac{45}{2}$ (2)

$\alpha^2 + 2\alpha\beta = -6$ (3)

from (1) $\beta = \frac{7}{2} - 2\alpha$

Sub into (3).

$\alpha^2 + 2\alpha\left(\frac{7}{2} - 2\alpha\right) = -6$.

$\alpha^2 + 7\alpha - 4\alpha^2 = -6$

$-3\alpha^2 + 7\alpha = -6$

$3\alpha^2 - 7\alpha - 6 = 0$

$(3\alpha + 2)(\alpha - 3) = 0$.

$\therefore \alpha = 3$ or $-\frac{2}{3}$.

When $\alpha = 3$ $\quad \beta = -\frac{5}{2}$

$\alpha = -\frac{2}{3}$ $\quad \beta = \frac{29}{6}$.

Sub into (2)

$\alpha = -\frac{2}{3}$ and $\beta = \frac{29}{6}$ not a solution.

$\therefore \alpha = 3$ $\beta = -\frac{5}{2}$

Q3 cont.

c) $x^3 - 6x^2 + px + q$

$(x-3)(x+2)$.

$P(3) = 0$

$P(-2) = 0$.

$P(3) = 3^3 - 6(3)^2 + p(3) + q$

$= -27 + 3p + q$

$\therefore 3p + q - 27 = 0$. ①

$P(-2) = (-2)^3 - 6(-2)^2 + p(-2) + q$

$= -32 - 2p + q$

$\therefore \cancel{=} -2p + q - 32 = 0$ ②

Subtract ② from ①

$5p + 5 = 0$

$5p = -5$

$p = -1$.

$\therefore 3(-1) + q - 27 = 0$.

$-30 + q = 0$.

$q = 30$.

$\therefore P(x) = x^3 - 6x^2 - x + 30$.

d) Prove

$1^2 + 3^2 + \ldots + (2n-1)^2 = \frac{1}{3}n(2n-1)(2n+1)$

Prove true for $n=1$.

$LHS = 1^2 \cancel{+} = 1$.

$RHS = \frac{1}{3}(1)(2-1)(2+1)$

$= 1$.

$\therefore$ True for $n=1$.

Assume true for $n=k$

$\therefore 1^2 + 3^2 + \ldots + (2k-1)^2 = \frac{1}{3}k(2k-1)(2k+1)$

Prove true for $n=k+1$

$1^2 + 3^2 + \ldots + (2k-1)^2 + (2k+1)^2 = \frac{1}{3}(k+1)(2k+1)(2k+3)$

$LHS = 1^2 + 3^2 + \ldots + (2k-1)^2 + (2k+1)^2$

$= \frac{1}{3}k(2k-1)(2k+1) + (2k+1)^2$.

$= (2k+1)\left[\frac{1}{3}k(2k-1) + (2k+1)\right]$

$= (2k+1)\left[\frac{k(2k-1) + 3(2k+1)}{3}\right]$

$= \frac{1}{3}(2k+1)(2k^2 - k + 6k + 3)$

$= \frac{1}{3}(2k+1)(2k^2 + 5k + 3)$

$= \frac{1}{3}(2k+1)(2k+3)(k+1)$

$= \frac{1}{3}(k+1)(2k+1)(2k+3)$

$= RHS$

$\therefore$ If statement true for $n=k$, it is also true for $n=k+1$.

$\therefore$ By the principal of mathematica induction true for all $n \geq 1$.

Question 4

a) $x - 2y = 6$ $\quad$ $4x - y = 1.$

$y = \frac{x}{2} - 3$ $\quad$ $y = 4x - 1.$

$$\tan\theta = \left| \frac{\frac{1}{2} - 4}{1 + (\frac{1}{2})(4)} \right|$$

$$= \left| \frac{-\frac{7}{2}}{3} \right|$$

$$= \left| \frac{-7}{6} \right| \quad = \frac{7}{6}.$$

$\therefore \tan\theta = \frac{7}{6}.$

$\therefore \theta = 49°24'.$

b) $\frac{1 - \cos 2\theta}{\sin 2\theta} = \tan\theta$

$$LHS = \frac{1 - \cos 2\theta}{\sin 2\theta}$$

$$= \frac{1 - (1 - 2\sin^2\theta)}{\sin 2\theta}$$

$$= \frac{2\sin^2\theta}{\sin 2\theta}$$

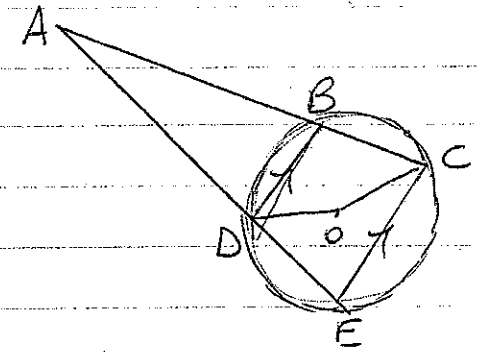
$$= \frac{\cancel{2}\sin^{\cancel{2}}\theta}{\cancel{2}\cancel{\sin\theta}\cos\theta}$$

$$= \frac{\sin\theta}{\cos\theta}$$

$= \tan\theta$

$= RHS.$

c)



i) Prove $AC = AE$.

$\angle ABD = \angle BCE$ (corresponding $\angle$, $BD \parallel CE$)

$\angle ABD = \angle CED$ (exterior $\angle$ of a cyclic quadrilateral).

$\therefore \angle BCE = \angle CED$

$\therefore \triangle ACE$ is an isosceles $\triangle$

$\therefore AC = AE$ (sides opposite equal angles in isosceles $\triangle$ are equal)

iii) Prove ACOD is cyclic.

$\angle DOC = 2 \times \angle DEC$ (angles on the same arc subtended at the circumference).

$\therefore \angle DOC = \angle DEC + \angle ECB.$ (angles in isosceles $\triangle$ from (i)).

$\therefore \angle DEC + \angle ECB + \angle CAD = 180°$ (angle sum of a $\triangle$).

$\therefore \angle DOC + \angle CAD = 180°$

$\therefore$ ACOD is a cyclic quadrilateral (Opposite angles are supplementary).

Question 5.

a) $P(2ap, ap^2)$ $Q(2aq, aq^2)$

$x^2 = 4ay.$

i) $m_{PQ} = \dfrac{ap^2 - aq^2}{2ap - 2aq}$

$= \dfrac{a(p^2 - q^2)}{2a(p-q)}$

$= \dfrac{p+q}{2}$

Eqn_{PQ} : $y - ap^2 = \left(\dfrac{p+q}{2}\right)(x - 2ap)$

$y - ap^2 = \left(\dfrac{p+q}{2}\right)x - 2ap\left(\dfrac{p+q}{2}\right)$

$y \cancel{- ap^2} = \left(\dfrac{p+q}{2}\right)x - \cancel{ap^2} - apq$

$y = \left(\dfrac{p+q}{2}\right)x - apq.$

When $R(2a, 3a)$.

$3a = \left(\dfrac{p+q}{2}\right)2a - apq.$

$3a = a(p+q) - apq$

$3 = (p+q) - pq$

$\therefore pq = p+q-3.$

$p+q = pq+3.$ (rearranges)

ii) $M_{PQ} = \left[\dfrac{2ap+2aq}{2}, \dfrac{ap^2+aq^2}{2}\right].$

$= \left[\dfrac{2a(p+q)}{2}, \dfrac{a(p^2+q^2)}{2}\right]$

$= \left[a(p+q), \dfrac{a(p^2+q^2)}{2}\right].$

$= \left[a(pq+3), \dfrac{a}{2}(p^2+q^2)\right]$

$= \left[a(pq+3), \dfrac{a}{2}\left((p+q)^2 - 2pq\right)\right]$

$= \left[a(pq+3), \dfrac{a}{2}\left((pq+3)^2 - 2pq\right)\right]$

iii) Let $x = a(pq+3) \Rightarrow pq = \dfrac{x}{a} - 3$

$y = \dfrac{a}{2}(pq+3)^2 - 2pq \Rightarrow \dfrac{2y}{a} + 2pq = (pq+3)^2$

$x^2 = a^2(pq+3)^2$

$x^2 = a^2\left(2pq + \dfrac{2y}{a}\right)$

$x^2 = a^2\left[\left(\dfrac{2y}{a}\right) + 2\left(\dfrac{x}{a} - 3\right)\right]$

$x^2 = 2ay + 2ax - 6a^2$

$x^2 - 2ax = 2ay - 6a^2.$

$\therefore x^2 - 2ax = 2a(y - 3a).$

b) Balloon travels $\frac{950}{3}$ m

$\frac{1}{\tan 78^\circ} = \frac{x}{3000}$.

$\therefore x = \frac{3000}{\tan 78^\circ}$

$\frac{1}{\tan 73^\circ} = \frac{y}{3000}$.

$\therefore y = \frac{3000}{\tan 73^\circ}$.

$$\cos\theta = \frac{\left(\frac{3000}{\tan 78^\circ}\right)^2 + \left(\frac{3000}{\tan 73^\circ}\right)^2 - \left(\frac{950}{3}\right)^2}{2\left(\frac{3000}{\tan 78^\circ}\right)\left(\frac{3000}{\tan 73^\circ}\right)}$$

$\cos\theta = \cancel{0.918}\ 0.981068\ldots$

$\theta = 11.16$.

$\therefore \theta = 11^\circ 10'$

c) $\sqrt{3}\sin x - \cos x = r\sin(x-\alpha)$

$\sqrt{3}\sin x - \cos x = r(\sin x\cos\alpha - \cos x\sin\alpha)$

$= r\cos\alpha\sin x - r\sin\alpha\cos x$

$\therefore r\cos\alpha = \sqrt{3}$ --- (1)

$r\sin\alpha = +1$. (2)

$r^2(\cos^2\alpha + \sin^2\alpha) = 4$.

$r^2 = 4$.

$r = 2$.

$\therefore \cos\alpha = \frac{\sqrt{3}}{2}$ and $\sin\alpha = \frac{+1}{2}$

α is in 1st quad.

$\therefore \tan\alpha = \frac{1}{\sqrt{3}}$

$\therefore \alpha = 30^\circ$.

$\sqrt{3}\sin x - \cos x = 2\sin(x - 30^\circ)$.

d) $4\sin\theta - 3\cos\theta = 2$.

$4\left(\frac{2t}{1+t^2}\right) - 3\left(\frac{1-t^2}{1+t^2}\right) = 2$.

$8t - 3 + 3t^2 = 2(1+t^2)$.

$3t^2 + 8t - 3 = 2 + 2t^2$.

$t^2 + 8t - 5 = 0$.

$t = \frac{-8 \pm \sqrt{8^2 - 4(1)(-5}}{2}$

$\therefore t = 0.58$ or -8.58

$\tan\frac{\theta}{2} = 0.58$ or $\tan\frac{\theta}{2} = -8.5$

$\frac{\theta}{2} = 30^\circ 7', 210^\circ 7', 96^\circ 39', 276^\circ 3$

$\therefore \theta = 60^\circ 14', 193^\circ 18'$.

Check $\theta = 180^\circ$.

$4\sin 180^\circ - 3\cos 180^\circ = 2$.

$0 + 3 = 2$.

$3 \neq 2$

$\therefore \theta = 180^\circ$ is not a solution